Date

UNIT - 2 : LIMITS AND CONTINUITY

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limit

Definition :

Let $f : S \rightarrow R$ be a function and 'a' be a limit point of 'S'. Here, A real number 'L' is called limit point of $f(x)$ at $x = a$, if $\forall \epsilon > 0 \ \exists \ \delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$, it is denoted by $\underset{x \to a}{Lt} f(x) = L$

(2) Let $f : S \rightarrow R$ be a function and 'a' be a limit point of 'S'. Here, A real number 'L' is called left hand limit point of $f(x)$ at $x = a$; if $\forall \epsilon > 0 \ \exists \ \delta > 0$ such that $a - \delta < x < a \Rightarrow |f(x) - L| < \epsilon$, it is denoted by $\underset{x \to a^-}{Lt} f(x) = L$

(3) Let $f : S \rightarrow R$ be a function and 'a' be a limit point of 'S'. A real number 'L' is called right hand limit point of $f(x)$ at $x = a$, if $\forall \epsilon > 0 \ \exists \ \delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - L| < \epsilon$, it is denoted by $\underset{x \to a^+}{Lt} f(x) = L$

Note :

1) $|x - a| < \delta \Longleftrightarrow a - \delta < x < a + \delta$

2) $|x| = -x, \ x < 0$
$\quad\;\; = x, \ x > 0$

3) $\underset{x \to a}{Lt} f(x) = l \Longleftrightarrow \underset{x \to a^-}{Lt} f(x) = \underset{x \to a^+}{Lt} f(x) = l$

Problems :

1) If $f : R \rightarrow R$ is a function defined by $f(x) = \begin{cases} 3x - 2, & x < 1 \\ 4x^2 - 3x, & x > 1 \end{cases}$.

then prove that $\underset{x \to 1}{Lt} f(x) = 1$

Sol:

$LHL = \underset{x \to 1^-}{Lt} f(x)$

$= \underset{x \to 1^-}{Lt} (3x-2)$

$= 3(1) - 2$

$= 1$

$RHL = \underset{x \to 1^+}{Lt} f(x)$

$= \underset{x \to 1^+}{Lt} (4x^2 - 3x)$

$= 4(1) - 3(1)$

$= 1$

$\therefore LHS = RHS = 1$

$\therefore \underset{x \to 1}{Lt} f(x) = 1$

2) If $f: R \to R$ is a function defined by $f(x) = \begin{cases} x, & x < \frac{1}{2} \\ x-1, & x > \frac{1}{2} \end{cases}$ then prove that $\underset{x \to 1/2}{Lt} f(x)$ is not exists

Sol:

$LHL = \underset{x \to 1/2^-}{Lt} f(x)$

$= \underset{x \to \frac{1}{2}^-}{Lt} (x)$

$= 1/2$

$RHL = \underset{x \to \frac{1}{2}^+}{Lt} f(x)$

$= \underset{x \to \frac{1}{2}^+}{Lt} (x-1)$

$= \frac{1}{2} - 1 = -1/2$

$\therefore LHL \neq RHL$

$\therefore \underset{x \to \frac{1}{2}}{Lt} f(x)$ is not exist

Imp

3) If $f: R \to R$ is a function defined by $f(x) = \begin{cases} x+2, & x<1 \\ 4x-1, & 1<x<3 \\ x^2+5, & x>3 \end{cases}$ then examine $\underset{x \to 1}{Lt} f(x)$ and $\underset{x \to 3}{Lt} f(x)$

Sol:

Examine $\underset{x \to 1}{Lt} f(x)$

$LHL = \underset{x \to 1^-}{Lt} f(x)$

$= \underset{x \to 1^-}{Lt} (x+2)$

$= 1+2$

$= 3$

$RHL = \underset{x \to 1^+}{Lt} f(x)$

$= \underset{x \to 1^+}{Lt} (4x-1)$

$= 4(1) - 1$

$= 3$

$\therefore LHL = RHL = 3$

$\therefore \underset{x \to 1}{Lt} f(x) = 3$

Examine $\underset{x \to 3}{Lt} f(x)$

$LHL = \underset{x \to 3^-}{Lt} f(x)$

$= \underset{x \to 3^-}{Lt} (4x-1)$

$= 4(3)-1$

$= 11$

$RHL = \underset{x \to 3^+}{Lt} f(x)$

$= \underset{x \to 3^+}{Lt} x^2+5$

$= 9+5$

$= 14$

$\therefore LHL \neq RHL$

$\therefore \underset{x \to 3}{Lt} f(x)$ is not exist

Imp

4) prove that $\underset{x \to 0}{Lt} \frac{|x|}{x}$ does not exist

sol: We know that,

$|x| = -x, x < 0$

$= x, x > 0$

$LHL = \underset{x \to 0^-}{Lt} f(x)$

$= \underset{x \to 0^-}{Lt} \frac{|x|}{x}$

$= \frac{-x}{x}$

$= -1$

$RHL = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x \to 0^+}{Lt} \frac{|x|}{x}$

$= \frac{x}{x}$

$= 1$

$\therefore LHL \neq RHL$

$\therefore \underset{x \to 0}{Lt} f(x)$ does not exist

5) prove that $\underset{x \to 0}{Lt} \frac{3|x|-2x}{3x-2|x|}$ does not exist

sol: We know that

$|x| = -x, x < 0$

$= x, x > 0$

$LHS = \underset{x \to 0^-}{Lt} f(x)$ $RHS = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x\to 0^-}{Lt} \dfrac{3|x|-2x}{3x-2|x|}$ $\qquad = \underset{x\to 0^+}{Lt} \dfrac{3|x|-2x}{3x-2|x|}$

$= \dfrac{-3x-2x}{3x+2x}$ $\qquad = \dfrac{3x-2x}{3x-2x}$

$= \dfrac{-5x}{5x}$ $\qquad = \dfrac{x}{x}$

$= -1$ $\qquad = 1$

$\therefore$ LHL $\neq$ RHL

$\therefore \underset{x\to 0}{Lt} f(x)$ does not exist

6) P.T $\underset{x\to 0}{Lt} \dfrac{3x+|x|}{7x-5|x|}$ does not exist

Sol: We know that,

$|x| = -x,\ x<0$

$= x,\ x>0$

LHL $= \underset{x\to 0^-}{Lt} f(x)$

$= \underset{x\to 0^-}{Lt} \dfrac{3x+|x|}{7x-5|x|}$

$= \dfrac{3x-x}{7x+5x}$

$= \dfrac{2x}{12x}$

$= \dfrac{1}{6}$

RHL $= \underset{x\to 0^+}{Lt} f(x)$

$= \underset{x\to 0^+}{Lt} \dfrac{3x+|x|}{7x-5|x|}$

$= \dfrac{3x+x}{7x-5x}$

$= \dfrac{4x}{2x}$

$= 2$

$\therefore$ LHL $\neq$ RHL

$\therefore \underset{x\to 0}{Lt} f(x)$ does not exist

7) P.T $\underset{x\to 0}{Lt} \dfrac{e^{1/x}}{1+e^{1/x}}$ does not exist

Sol: LHL $= \underset{x\to 0^-}{Lt} f(x)$

$= \underset{x\to 0^-}{Lt} \dfrac{e^{1/x}}{1+e^{1/x}}$ $\boxed{\begin{array}{l} e^{-\infty}=0 \\ e^{\infty}=\infty \end{array}}$

$x\to 0^-$ means $x = 0-h$ as $h\to 0$

RHL $= \underset{x\to 0^+}{Lt} f(x)$

$= \underset{x\to 0^+}{Lt} \dfrac{e^{1/x}}{1+e^{1/x}}$

$x\to 0^+$ means $x = 0+h$ as $h\to 0$

$$= \underset{h\to 0}{Lt} \frac{e^{\frac{1}{0-h}}}{1+e^{1/0-h}}$$

$$= \underset{h\to 0}{Lt} \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \frac{e^{-1/0}}{1+e^{-1/0}}$$

$$= \frac{e^{-\infty}}{1+e^{-\infty}}$$

$$= \frac{0}{1+0}$$

$$= 0$$

$$= \underset{h\to 0}{Lt} \frac{e^{1/0+h}}{1+e^{1/0+h}}$$

$$= \underset{h\to 0}{Lt} \frac{e^{1/h}}{1+e^{1/h}}$$

$$= \underset{h\to 0}{Lt} \frac{e^{1/h}}{e^{1/h}\left(\frac{1}{e^{1/h}}+1\right)}$$

$$= \underset{h\to 0}{Lt} \frac{1}{e^{-1/h}+1}$$

$$= \frac{1}{e^{-1/0}+1}$$

$$= \frac{1}{e^{-\infty}+1} = \frac{1}{0+1} = 1$$

$\therefore$ LHL $\neq$ RHL

$\therefore \underset{x\to 0}{Lt} f(x)$ does not exist.

Imp

8) P.T $\underset{x\to 0}{Lt} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$ does not exist

Sol: LHL $= \underset{x\to 0^-}{Lt} f(x)$

$$= \underset{x\to 0^-}{Lt} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$$

$x\to 0^-$ means $x = 0-h$ as $h\to 0$

$$= \underset{h\to 0}{Lt} \frac{e^{\frac{1}{0-h}}-e^{-\frac{1}{0-h}}}{e^{\frac{1}{0-h}}+e^{-\frac{1}{0-h}}}$$

$$= \underset{h\to 0}{Lt} \frac{e^{-\frac{1}{h}}-e^{\frac{1}{h}}}{e^{-\frac{1}{h}}+e^{\frac{1}{h}}}$$

RHL $= \underset{x\to 0^+}{Lt} f(x)$

$$= \underset{x\to 0^+}{Lt} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$$

$x\to 0^+$ means $x = 0+h$ as $h\to 0$

$$= \underset{h\to 0}{Lt} \frac{e^{\frac{1}{0+h}}-e^{-\frac{1}{0+h}}}{e^{\frac{1}{0+h}}+e^{-\frac{1}{0+h}}}$$

$$= \underset{h\to 0}{Lt} \frac{e^{\frac{1}{h}}-e^{-\frac{1}{h}}}{e^{\frac{1}{h}}+e^{-\frac{1}{h}}}$$

$$= \lim_{h\to 0} \frac{e^{-\frac{1}{h}}\left[\frac{e^{-1/h}}{e^{1/h}}-1\right]}{e^{-\frac{1}{h}}\left[\frac{e^{-1/h}}{e^{1/h}}+1\right]}$$

$$= \lim_{h\to 0}\left[\frac{e^{-1/h}\,e^{-1/h}-1}{e^{-1/h}\,e^{-1/h}+1}\right]$$

$$= \lim_{h\to 0}\left[\frac{e^{-2/h}-1}{e^{-2/h}+1}\right]$$

$$= \frac{e^{-2/0}-1}{e^{-2/0}+1}$$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1} = \frac{-1}{1} = -1$$

$$= \lim_{h\to 0} \frac{e^{1/h}\left[1-\frac{e^{-1/h}}{e^{1/h}}\right]}{e^{1/h}\left[1+\frac{e^{-1/h}}{e^{1/h}}\right]}$$

$$= \lim_{h\to 0}\left[\frac{1-e^{-1/h}.e^{-1/h}}{1+e^{-1/h}.e^{-1/h}}\right]$$

$$= \lim_{h\to 0}\left[\frac{1-e^{-2/h}}{1+e^{-2/h}}\right]$$

$$= \frac{1-e^{-2/0}}{1+e^{-2/0}}$$

$$= \frac{1-e^{-\infty}}{1+e^{-\infty}} = \frac{1-0}{1+0} = \frac{1}{1} = 1$$

$\therefore$ LHL $\neq$ RHL

$\therefore \lim_{h\to 0} f(x)$ does not exist.

9) prove that $\lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

Sol: LHL $= \lim_{x\to 0^-} f(x)$

$$= \lim_{x\to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{0-h}}-1}{e^{\frac{1}{0-h}}+1}$$

$$= \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

RHL $= \lim_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$x \to 0^+$ means $x = 0+h$ as $h \to 0$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{0+h}}-1}{e^{\frac{1}{0+h}}+1}$$

$$= \lim_{h\to 0} \frac{e^{1/h}-1}{e^{1/h}+1}$$

$$= \lim_{h\to 0} \frac{e^{1/h}\left[1-\frac{1}{e^{1/h}}\right]}{e^{1/h}\left[1+1/e^{1/h}\right]}$$

$= \dfrac{e^{-1/0} - 1}{e^{-1/0} + 1} \Rightarrow \dfrac{e^{-\infty} - 1}{e^{-\infty} + 1}$

$= \dfrac{0-1}{0+1} = \dfrac{-1}{1} = -1$

$= \underset{h \to 0}{Lt} \dfrac{1 - e^{-1/h}}{1 + e^{-1/h}}$

$= \dfrac{1 - e^{-1/0}}{1 + e^{-1/0}}$

$= \dfrac{1 - e^{-\infty}}{1 + e^{-\infty}} = \dfrac{1-0}{1+0} = \dfrac{1}{1} = 1$

$LHL \neq RHL$

$\therefore \underset{x \to 0}{Lt} f(x)$ does not exist.

10) prove that $\underset{x \to 0}{Lt} \dfrac{e^{1/x} + 1}{e^{\frac{1}{x}} - 1}$ does not exist.

sol: $LHL = \underset{x \to 0^-}{Lt} f(x)$

$= \underset{x \to 0^-}{Lt} \dfrac{e^{\frac{1}{x}} + 1}{e^{\frac{1}{x}} - 1}$

$x \to 0^-$ means $x = 0 - h$ as $h \to 0$

$= \underset{h \to 0^-}{Lt} \dfrac{e^{\frac{1}{0-h}} + 1}{e^{\frac{1}{0-h}} - 1}$

$= \underset{h \to 0}{Lt} \dfrac{e^{-1/h} + 1}{e^{-1/h} - 1}$

$= \dfrac{e^{-1/0} + 1}{e^{-1/0} - 1} \Rightarrow \dfrac{e^{-\infty} + 1}{e^{-\infty} - 1}$

$= \dfrac{0+1}{0-1} = \dfrac{1}{-1} = -1$

$RHL = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x \to 0^+}{Lt} \dfrac{e^{1/x} + 1}{e^{\frac{1}{x}} - 1}$

$x \to 0^+$ means $x = 0 + h$ as $h \to 0$

$= \underset{h \to 0}{Lt} \dfrac{e^{\frac{1}{0+h}} + 1}{e^{\frac{1}{0+h}} - 1}$

$= \underset{h \to 0}{Lt} \dfrac{e^{1/h} + 1}{e^{\frac{1}{h}} - 1}$

$= \underset{h \to 0}{Lt} \dfrac{e^{1/h}\left[1 + \frac{1}{e^{1/h}}\right]}{e^{1/h}\left[1 - \frac{1}{e^{1/h}}\right]}$

$= \underset{h \to 0}{Lt} \dfrac{1 + e^{-1/h}}{1 - e^{-1/h}}$

$= \dfrac{1 + e^{-1/0}}{1 - e^{-1/0}} = \dfrac{1 + e^{-\infty}}{1 - e^{-\infty}}$

$= \dfrac{1+0}{1-0} = \dfrac{1}{1} = 1$

$\therefore LHL \neq RHL$

$\therefore \underset{x \to 0}{Lt} f(x)$ does not exist.

11) Let $S = R - \{0\}$, define $f: S \to R$ such that $f(x) = x \sin\frac{1}{x}$. prove that $\underset{x \to 0}{Lt} f(x) = 0$

sol:

$LHL = \underset{x \to 0^-}{Lt} f(x)$

$= \underset{x \to 0^-}{Lt} x \sin\frac{1}{x}$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$= \underset{h \to 0}{Lt} (0-h) \sin\left[\frac{1}{0-h}\right]$

$= \underset{h \to 0}{Lt} (-h) \sin\left[\frac{1}{-h}\right]$

$= \underset{h \to 0}{Lt} (-h) \left(-\sin\frac{1}{h}\right)$

$= \underset{h \to 0}{Lt} h \sin\frac{1}{h}$

$= 0 \quad [\because 1 \leq \sin \leq 1]$

$RHL = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x \to 0^+}{Lt} x \sin\frac{1}{x}$

$x \to 0^+$ means $x = 0+h$ as $h \to 0$

$= \underset{h \to 0}{Lt} (0+h) \sin\left[\frac{1}{0+h}\right]$

$= \underset{h \to 0}{Lt} (h) \sin\left[\frac{1}{h}\right]$

$= 0$

$\therefore LHL = RHL$

$\therefore \underset{x \to 0}{Lt} f(x) = 0$

12) Let $S = R - \{0\}$, defined $f: S \to R$ such that $f(x) = x \cos\frac{1}{x}$ prove that $\underset{x \to 0}{Lt} f(x) = 0$

sol:

$LHL = \underset{x \to 0^-}{Lt} f(x)$

$= \underset{x \to 0^-}{Lt} x \cos\frac{1}{x}$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$= \underset{h \to 0}{Lt} (0-h) \cos\left[\frac{1}{0-h}\right]$

$= \underset{h \to 0}{Lt} (-h) \cos\left[\frac{1}{-h}\right]$

$= \underset{h \to 0}{Lt} (-h) \left(\cos\frac{1}{h}\right)$

$= \underset{h \to 0}{Lt} -h \cos\frac{1}{h}$

$(\because -1 \leq \cos x \leq 1)$

$RHL = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x \to 0^+}{Lt} x \cos\frac{1}{x}$

$x \to 0^+$ means $x = 0+h$ as $h \to 0$

$= \underset{h \to 0}{Lt} (0+h) \cos\left[\frac{1}{0+h}\right]$

$= \underset{h \to 0}{Lt} (h) \cos\left[\frac{1}{h}\right]$

$= 0$

$\therefore LHL = RHL$

$\therefore \underset{x \to 0}{Lt} f(x) = 0$

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Continuity : Let $f: S \to R$ be a function and $a \in S$ then we say that f is continuous at a, if $\forall \epsilon > 0 \; \exists \; \delta > 0$ such that $|x-a| < \delta \Rightarrow$ ~~$\underset{x \to a}{Lt} f(x) = f(a)$~~. $|f(x) - f(a)| < \epsilon$. It is denoted by $\underset{x \to a}{Lt} f(x) = f(a)$

Note : A Function which is not continuous is called discontinuous function.

Types of Discontinuous function :

* Removable discontinuous function :- A Function 'F' is said to be removable at a, if $\underset{x \to a}{Lt} f(x)$ exists and $\underset{x \to a}{Lt} f(x) \neq f(a)$.

* Jump discontinuous function :- A Function 'F' is said to be jump discontinuous function at a, if $\underset{x \to a^-}{Lt} f(x)$ and $\underset{x \to a^+}{Lt} f(x)$ exists and they are not equal.

1) Examine the continuity of $f(x) = \begin{cases} 2x, & 0 \le x < 1 \\ 3, & x = 1 \\ 4x, & 1 < x \le 2 \end{cases}$ at $x = 1$

Sol: $LHL = \underset{x \to 1^-}{Lt} f(x)$

$= \underset{x \to 1^-}{Lt} 2x$

$= 2(1) = 2$

$RHL = \underset{x \to 1^+}{Lt} f(x)$

$= \underset{x \to 1^+}{Lt} 4x$

$= 4(1)$

$= 4$

$LHL \neq RHL$

$\therefore f(x)$ is jump discontinuous function.

2) Examine the continuity of $f(x) = \begin{cases} \frac{x^2}{2}, & 0 \le x < 1 \\ 2x^2 - 2x + \frac{3}{2}, & 1 < x \le 2 \end{cases}$ at $x = 1$

Sol: $LHL = \underset{x \to 1^-}{Lt} f(x)$

$= \underset{x \to 1^-}{Lt} \frac{x^2}{2}$

$= 1/2$

$RHL = \underset{x \to 1^+}{Lt} f(x)$

$= \underset{x \to 1^+}{Lt} 2x^2 - 2x + \frac{3}{2}$

$= 2(1)^2 - 2(1) + \frac{3}{2}$

$= 3/2$

S.L.V.

$\therefore$ LHL $\neq$ RHL

$\therefore f(x)$ is jump ~~distribution~~ discontinuous function.

3) Examine the continuity of $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ if $x \neq 0$ and $f(0) = 1$ at $x = 0$

sol: $LHL = \underset{x \to 0^-}{Lt} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

$x \to 0^-$ means $x = 0 - h$ as $h \to 0$

$$= \underset{h \to 0}{Lt} \frac{e^{\frac{1}{0-h}} - e^{\frac{-1}{0-h}}}{e^{\frac{1}{0-h}} + e^{\frac{-1}{0-h}}}$$

$$= \underset{h \to 0}{Lt} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \underset{h \to 0}{Lt} \frac{e^{1/h}\left[\frac{e^{-1/h}}{e^{1/h}} - 1\right]}{e^{1/h}\left[\frac{e^{-1/h}}{e^{1/h}} + 1\right]}$$

$$= \underset{h \to 0}{Lt} \frac{e^{-2/h} - 1}{e^{-2/h} + 1}$$

$$= \frac{e^{-\infty} - 1}{e^{-\infty} + 1}$$

$$= \frac{0 - 1}{0 + 1}$$

$$= -1$$

$RHL = \underset{x \to 0^+}{Lt} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

$x \to 0^+$ means $x = 0 + h$ as $h \to 0$

$$= \underset{h \to 0}{Lt} \frac{e^{\frac{1}{0+h}} - e^{\frac{-1}{0+h}}}{e^{\frac{1}{0+h}} + e^{\frac{-1}{0+h}}}$$

$$= \underset{h \to 0}{Lt} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$

$$= \underset{h \to 0}{Lt} \frac{e^{1/h}\left[1 - e^{-2/h}\right]}{e^{1/h}\left[1 + e^{-2/h}\right]}$$

$$= \frac{1 - e^{-\infty}}{1 + e^{-\infty}}$$

$$= \frac{1 - 0}{1 + 0}$$

$$= 1$$

$\therefore$ LHL $\neq$ RHL

$\therefore f(x)$ is jump discontinuous function.

4) Examine the continuity of $f(x) = \dfrac{e^{1/x}}{1 + e^{1/x}}$ if $x \neq 0$ and $f(0) = 0$ at $x = 0$

sol: $LHL = \underset{x \to 0^-}{Lt} \dfrac{e^{1/x}}{1 + e^{1/x}}$

$x \to 0^-$ means $x = 0 - h$ as $h \to 0$

$$= \underset{h \to 0}{Lt} \frac{e^{\frac{1}{0-h}}}{1 + e^{\frac{1}{0-h}}}$$

$RHL = \underset{x \to 0^+}{Lt} \dfrac{e^{1/x}}{1 + e^{1/x}}$

$x \to 0^+$ means $x = 0 + h$ as $h \to 0$

$$= \underset{h \to 0}{Lt} \frac{e^{\frac{1}{0+h}}}{1 + e^{\frac{1}{0+h}}}$$

$= \underset{h\to 0}{Lt} \dfrac{e^{-1/h}}{1+e^{-1/h}}$

$= \dfrac{e^{-\infty}}{1+e^{-\infty}}$

$= \dfrac{0}{1+0}$

$= 0$

$= \underset{h\to 0}{Lt} \dfrac{e^{1/h}}{1+e^{1/h}}$

$= \underset{h\to 0}{Lt} \dfrac{e^{1/h}}{e^{1/h}(e^{-1/h}+1)}$

$= \underset{h\to 0}{Lt} \dfrac{1}{e^{-1/h}+1}$

$= \dfrac{1}{0+1}$

$= 1$

$LHL \neq RHL$

$\therefore f(x)$ is jump discontinuous functions.

5) Examine the continuity of $f(x) = x\left[\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right]$ if $x \neq 0$ and $f(0)=1$ at $x=0$

$LHL = \underset{x\to 0^-}{Lt} f(x)$

$= \underset{x\to 0^-}{Lt} x\left[\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right]$

$x \to 0^-$ means $x = 0-h$ as $h \to 0$

$= \underset{h\to 0}{Lt} (0-h)\left(\dfrac{e^{\frac{1}{0-h}}-e^{\frac{-1}{0-h}}}{e^{\frac{1}{0-h}}+e^{\frac{-1}{0-h}}}\right)$

$= \underset{h\to 0}{Lt} (-h)\left[\dfrac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}\right]$

$= \underset{h\to 0}{Lt} (-h) \dfrac{e^{1/h}(e^{-2/h}-1)}{e^{1/h}(e^{-2/h}+1)}$

$= -(0)\left[\dfrac{e^{-\infty}-1}{e^{-\infty}+1}\right]$

$= \dfrac{0(0+1)}{0+1}$

$= 0$

$RHL = \underset{x\to 0^+}{Lt} f(x)$

$= \underset{x\to 0^+}{Lt} x\left[\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right]$

$x \to 0^+$ means $x = 0+h$ as $h \to 0$

$= \underset{h\to 0}{Lt} (0+h)\left[\dfrac{e^{\frac{1}{0+h}}-e^{\frac{-1}{0+h}}}{e^{\frac{1}{0+h}}+e^{\frac{-1}{0+h}}}\right]$

$= \underset{h\to 0}{Lt} (h)\left[\dfrac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}\right]$

$= \underset{h\to 0}{Lt} (h)\left[\dfrac{e^{1/h}(1-e^{-1/h})}{e^{1/h}(1+e^{-1/h})}\right]$

$= 0\left[\dfrac{1-e^{-\infty}}{1+e^{-\infty}}\right]$

$= 0\left(\dfrac{1-0}{1+0}\right)$

$= 0(1)$

$= 0$

$\because LHL = RHL$

$\therefore \underset{x \to 0}{Lt} f(x) = 0$

But $f(0) = 1$ (given)

$\therefore \underset{x \to 0}{Lt} f(x) \neq f(0)$

$\therefore f(x)$ is removable discontinuous function.

6) Determine the constants a and b so that the function $f(x)$ is defined by $f(x) = \begin{cases} 2x+1, & x \leq 1 \\ ax^2+b, & 1 < x < 3 \\ 5x+2a, & x \geq 3 \end{cases}$ is continuous everywhere

Sol:

Continuity at $x = 1$

$LHL = \underset{x \to 1^-}{Lt} f(x)$

$= \underset{x \to 1^-}{Lt} (2x+1)$

$= 2(1)+1$

$= 3$

$RHL = \underset{x \to 1^+}{Lt} f(x)$

$= \underset{x \to 1^+}{Lt} ax^2+b$

$= a(1)+b$

$= a+b$

$f(x)$ is continuity at $x=1 \Rightarrow LHL = RHL$

ie; $a+b = 3 \to (1)$

Continuity at $x = 3$

$LHL = \underset{x \to 3^-}{Lt} f(x)$

$= \underset{x \to 3^-}{Lt} ax^2+b$

$= a(3)^2+b$

$= 9a+b$

$\because RHL = \underset{x \to 3^+}{Lt} f(x)$

$= \underset{x \to 3^+}{Lt} 5x+2a$

$= 5(3)+2a$

$= 15+2a$

Since $f(x)$ is continuous at $x=3$

$\Rightarrow LHL = RHL$

ie, $9a+b = 15+2a$

$7a+b = 15 \to (2)$

solving ① & ②

$a+b=3$

$7a+b=15$

$6a=12$

$a=12/6$

$\boxed{a=2}$

$\boxed{\therefore b=1}$

$2+b=3$

$b=1$

7) Examine the continuity of $f(x)=\dfrac{1-\cos 2x}{1-\cos 4x}$, if $x\neq 0$ and $f(0)=\frac{2}{3}$ at $x=0$

Sol: $LHL = \underset{x\to 0^-}{Lt} f(x)$

$= \underset{x\to 0^-}{Lt}\left[\dfrac{1-\cos 2x}{1-\cos 4x}\right]$

$= \underset{x\to 0^-}{Lt}\left[\dfrac{0-(-2\sin 2x)}{0-(-4\sin 4x)}\right]$

$= \underset{x\to 0^-}{Lt}\left[\dfrac{2\sin 2x}{4\sin 4x}\right]$

$= \frac{1}{2}\underset{x\to 0^-}{Lt}\dfrac{\sin 2x}{\sin 4x}$

$= \frac{1}{2}\underset{x\to 0^-}{Lt}\dfrac{+2\cos 2x}{+4\cos 4x}$

$= \frac{1}{4}\underset{x\to 0^-}{Lt}\dfrac{\cos 2x}{\cos 4x}$

$= \frac{1}{4}\left[\dfrac{\cos 0}{\cos 0}\right]$

$= \frac{1}{4}\left[\frac{1}{1}\right]$

$= 1/4$

$RHL = \underset{x\to 0^+}{Lt} f(x)$

$= \underset{x\to 0^+}{Lt}\left[\dfrac{1-\cos 2x}{1-\cos 4x}\right]$

$= \underset{x\to 0^+}{Lt}\left[\dfrac{0-\cos 2x}{0-\cos 4x}\right]$

$= \underset{x\to 0^+}{Lt}\left[\dfrac{0-(-2\sin 2x)}{0-(-4\sin 4x)}\right]$

$= \underset{x\to 0^+}{Lt}\left[\dfrac{2\sin 2x}{4\sin 4x}\right]$

$= \frac{1}{2}\underset{x\to 0^+}{Lt}\left[\dfrac{\sin 2x}{\sin 4x}\right]$

$= \frac{1}{2}\underset{x\to 0^+}{Lt}\left[\dfrac{2\cos 2x}{4\cos 4x}\right]$

$= \frac{1}{4}\underset{x\to 0^+}{Lt}\left[\dfrac{\cos 2x}{\cos 4x}\right]$

$= \frac{1}{4}\left[\dfrac{\cos 0}{\cos 0}\right]$

$= \frac{1}{4}\left[\frac{1}{1}\right]$

$= 1/4$

$LHL = RHL$

$\underset{x \to 0}{Lt} f(x) = \frac{1}{4}$

Given $f(0) = 2/2$

$\underset{x \to 0}{Lt} f(x) \neq f(0)$

$\left[\because \frac{1}{4} \neq 2/3\right]$

8) Examine the continity of $f(x) = \frac{1 - \cos x}{x^2}$, if $x \neq 0$ and $f(0) = 1$ at $x = 0$

Sol: $LHL = \underset{x \to 0^-}{Lt} f(x)$

$= \underset{x \to 0^-}{Lt} \frac{1 - \cos x}{x^2}$

$= \underset{x \to 0^-}{Lt} \frac{0 - (-\sin x)}{2x}$

$= \frac{1}{2} \underset{x \to 0^-}{Lt} \frac{\sin x}{x} \left[\because \underset{\theta \to 0}{Lt} \frac{\sin\theta}{\theta} = 1\right]$

$= \frac{1}{2}(1)$

$= \frac{1}{2}$

$RHL = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x \to 0^+}{Lt} \frac{1 - \cos x}{x^2}$

$= \underset{x \to 0^+}{Lt} \left(\frac{0 - (-\sin x)}{2x}\right)$

$= \frac{1}{2} \underset{x \to 0^+}{Lt} \frac{\sin x}{x}$

$= \frac{1}{2}(1)$

$= \frac{1}{2}$

$LHL = RHL$

But $\underset{x \to 0}{Lt} f(x) \neq f(0)$

$\therefore f(x)$ is not continuous

Imp 9) Examine the contiity of $f(x) = |x| + |x-1|$ at $x = 0, 1$.

Sol: continuity at $x = 0$

$x < 0, x < 1$

If $x < 0 \Rightarrow |x| = -x$

If $x < 1 \Rightarrow |x-1| = -(x-1)$

$f(x) = -x - x + 1$

$= 1 - 2x$

$x > 0, x < 1$

If $x > 0 \Rightarrow |x| = x$

If $x < 1 \Rightarrow |x-1| = -(x-1)$

$f(x) = x - x + 1$

$= 1$

Also, $f(0) = |0| + |0-1|$

$= 1$

$$f(x) = \begin{cases} 1-2x, & x<0 \\ 1, & x>0 \\ 1, & x=0 \end{cases}$$

$LHL = \underset{x \to 0^-}{Lt} f(x)$

$= \underset{x \to 0^-}{Lt} (1-2x)$

$= 1-2(0)$

$= 1$

$RHL = \underset{x \to 0^+}{Lt} f(x)$

$= \underset{x \to 0^+}{Lt} (1)$

$= 1$

$\therefore LHL = RHL = 1$

$\therefore \underset{x \to 0}{Lt} f(x) = 1$, also $f(0) = 1$

$\therefore \underset{x \to 0}{Lt} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x=0$

<u>continuity at $x=1$</u>

$x<1, x>0$

If $x<1 \Rightarrow |x-1| = -(x-1)$

If $x>0 \Rightarrow |x| = x$

$f(x) = x-x+1$

$= 1$

$x>1, x>0$

If $x>1 \Rightarrow |x-1| = x-1$

If $x>0 \Rightarrow |x| = x$

$f(x) = x+x-1$

$= 2x-1$

Also, $f(1) = |1| + |1-1|$

$= 1$

$$\therefore f(x) = \begin{cases} 1, & x<1 \\ 2x-1, & x>1 \\ 1, & x=1 \end{cases}$$

$LHL = \underset{x \to 1^-}{Lt} f(x)$

$= \underset{x \to 1^-}{Lt} (1)$

$= 1$

$RHL = \underset{x \to 1^+}{Lt} f(x)$

$= \underset{x \to 1^+}{Lt} (2x-1)$

$= 2(1)-1$

$= 1$

$\therefore LHL = RHL = 1$

$\therefore \underset{x \to 1}{Lt}\ f(x) = 1$

$\therefore \underset{x \to 1}{Lt}\ f(x) = f(1)$

$\therefore f(x)$ is continuous at $x = 1$

10) Examine the continuity of $f(x) = |x-1| + |x-2|$ at $x = 1, 2$

Sol: <u>continuity at $x=1$</u>

$x < 1,\ x < 2$

If $x < 1 \Rightarrow |x-1| = -(x-1)$

If $x < 2 \Rightarrow |x-2| = -(x-2)$

$f(x) = -x+1-x+2$

$= 3-2x$

$x > 1,\ x < 2$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x < 2 \Rightarrow |x-2| = -(x-2)$

$f(x) = x-1-x+2$

$= 1$

Also, $f(1) = |1-1| + |1-2|$

$= 1$

$$f(x) = \begin{cases} 3-2x, & x < 1 \\ 1, & x > 1 \\ 1, & x = 1 \end{cases}$$

$LHL = \underset{x \to 1^-}{Lt}\ f(x)$

$= \underset{x \to 1^-}{Lt}\ 3-2x$

$= 3-2(1)$

$= 1$

$RHL = \underset{x \to 1^+}{Lt}\ f(x)$

$= \underset{x \to 1^+}{Lt}\ 1$

$= 1$

$\therefore LHL = RHL = 1$

$\therefore \underset{x \to 1}{Lt}\ f(x) = 1$, also $f(1) = 1$

$\therefore \underset{x \to 1}{Lt}\ f(x) = f(1)$

$\therefore f(x)$ is continuous at $x = 1$

<u>continuity at $x = 2$</u>

$x < 2,\ x > 1$

If $x < 2 \Rightarrow |x-2| = -(x-2)$

If $x > 1 \Rightarrow |x-1| = x-1$

$x > 2,\ x > 1$

If $x > 2 \Rightarrow |x-2| = x-2$

If $x > 1 \Rightarrow |x-1| = x-1$

$f(x) = x-1-x+2$
$\quad = 1$

$f(x) = x-1+x-2$
$\quad = 2x-3$

Also, $f(2) = |2-1| + |2-2|$
$\quad = 1$

$$f(x) = \begin{cases} 1, & x<2 \\ 2x-3, & x>2 \\ 1, & x=2 \end{cases}$$

$LHL = \underset{x \to 2^-}{Lt} f(x)$
$\quad = \underset{x \to 2^-}{Lt} 1$
$\quad = 1$

$RHL = \underset{x \to 2^+}{Lt} f(x)$
$\quad = \underset{x \to 2^+}{Lt} (2x-3)$
$\quad = 2(2)-3 = 1$

$\therefore LHL = RHL = 1$

$\therefore \underset{x \to 2}{Lt} f(x) = 1$, also $f(1) = 1$

$\therefore \underset{x \to 2}{Lt} f(x) = f(2)$

$\therefore f(x)$ is continuous at $x=2$

11) Examine the continuity of $f(x) = |x| + |x-1| + |x-2|$ at $x = 0, 1, 2$

Sol: Continuity at $x=0$

$x<0, x<1, x<2$

If $x<0 \Rightarrow |x| = -x$
If $x<1 \Rightarrow |x-1| = -(x-1)$
If $x<2 \Rightarrow |x-2| = -(x-2)$

$f(x) = -x-x+1-x+2$
$\quad = 3-3x$

$x>0, x<1, x<2$

If $x>0 \Rightarrow |x| = x$
If $x<1 \Rightarrow |x-1| = -(x-1)$
If $x<2 \Rightarrow |x-2| = -(x-2)$

$f(x) = x-x+1-x+2$
$\quad = 3-x$

Also, $f(0) = |0| + |0-1| + |0-2|$
$\quad = 1+2$
$\quad = 3$

$\therefore f(x) = \begin{cases} 3-3x, & x<0 \\ 3-x, & x>0 \\ 3, & x=0 \end{cases}$

$LHL = \underset{x\to 0^-}{Lt} f(x)$

$= \underset{x\to 0^-}{Lt} (3-3x)$

$= 3-3(0)$

$= 3$

$RHL = \underset{x\to 0^+}{Lt} f(x)$

$= \underset{x\to 0^+}{Lt} 3-x$

$= 3-0$

$= 3$

$\therefore LHL = RHL = 3$

$\underset{x\to 0}{Lt} f(x) = 3$, also $f(0) = 3$

$\underset{x\to 0}{Lt} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x=0$

<u>continuity at $x=1$</u>

$x<1,\ x<2,\ x>0$

If $x<1 \Rightarrow |x-1| = -(x-1)$

If $x<2 \Rightarrow |x-2| = -(x-2)$

If $x>0 \Rightarrow |x| = x$

$f(x) = x-x+1-x+2$

$= 3-x$

$x>1,\ x<2,\ x>0$

If $x>1 \Rightarrow |x-1| = x-1$

If $x<2 \Rightarrow |x-2| = -(x-2)$

If $x>0 \Rightarrow |x| = x$

$f(x) = x+x-1-x+2$

$= 1+x$

Also, $f(1) = |1| + |1-1| + |1-2|$

$= 1+1 = 2$

$\therefore f(x) = \begin{cases} 3-x, & x<1 \\ 1+x, & x>1 \\ 2, & x=1 \end{cases}$

$LHL = \underset{x\to 1^-}{Lt} f(x)$

$= \underset{x\to 1^-}{Lt} (3-x)$

$= 3-1$

$= 2$

$RHL = \underset{x\to 1^+}{Lt} f(x)$

$= \underset{x\to 1^+}{Lt} (1+x)$

$= 1+1$

$= 2$

$\therefore LHL = RHL = 2$

$\underset{x \to 0}{Lt} f(x) = 2$, also $f(1) = 2$

$\underset{x \to 0}{Lt} f(x) = f(1)$

$\therefore f(x)$ is continuous at $x = 1$

Continuity at $x = 2$

$x < 2, x > 1, x > 0$

If $x < 2 \Rightarrow |x-2| = -(x-2)$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x > 0 \Rightarrow |x| = x$

$f(x) = \cancel{x} + x - 1 - \cancel{x} + 2$

$= 1 + x$

$x > 2, x > 1, x > 0$

If $x > 2 \Rightarrow |x-2| = x-2$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x > 0 \Rightarrow |x| = x$

$f(x) = x + x - 1 + x - 2$

$= 3x - 3$

Also, $f(2) = |2| + |2-1| + |2-2|$

$= 2 + 1 + 0$

$= 3$

$$\therefore f(x) = \begin{cases} 1+x, & x < 2 \\ 3x-3, & x > 2 \\ 3, & x = 2 \end{cases}$$

$LHL = \underset{x \to 2^-}{Lt} f(x)$

$= \underset{x \to 2^-}{Lt} (1+x)$

$= 1 + 2$

$= 3$

$RHL = \underset{x \to 2^+}{Lt} f(x)$

$= \underset{x \to 2^+}{Lt} (3x-3)$

$= 3(2) - 3$

$= 3$

$\therefore LHL = RHL = 3$

$\therefore \underset{x \to 2}{Lt} f(x) = 3$, also $f(2) = 3$

$\underset{x \to 2}{Lt} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x = 2$

12) Examine the continuity of $f(x) = |x-1| + |x-2| + |x-3|$ at $x = 1, 2, 3$

Sol: Continuity at $x = 1$

If $x < 1$, $x < 2$, $x < 3$

If $x < 1 \Rightarrow |x-1| = -(x-1)$

If $x < 2 \Rightarrow |x-2| = -(x-2)$

If $x < 3 \Rightarrow |x-3| = -(x-3)$

$f(x) = -x+1-x+2-x+3$

$= 6-3x$

If $x > 1$, $x < 2$, $x < 3$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x < 2 \Rightarrow |x-2| = -(x-2)$

If $x < 3 \Rightarrow |x-3| = -(x-3)$

$f(x) = x-1-x+2-x+3$

$= 4-x$

$f(1) = |1-1| + |1-2| + |1-3|$

$= 3$

$LHL = \underset{x \to 1^-}{Lt} f(x)$

$= \underset{x \to 1^-}{Lt} (6-3x)$

$= 6-3(1)$

$= 3$

$RHL = \underset{x \to 1^+}{Lt} f(x)$

$= \underset{x \to 1}{Lt} (4-x)$

$= (4-1)$

$= 3$

$\therefore LHL = RHL$

$f(x) = f(1) = 3$

$\therefore f(x)$ is continuous at $x = 1$

Continuity at $x = 2$

If $x > 1$, $x < 2$, $x < 3$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x < 2 \Rightarrow |x-2| = -(x-2)$

If $x < 3 \Rightarrow |x-3| = -(x-3)$

$f(x) = x-1-x+2-x+3$

$= 4-x$

If $x > 1$, $x > 2$, $x < 3$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x > 2 \Rightarrow |x-2| = x-2$

If $x < 3 \Rightarrow |x-3| = -(x-3)$

$f(x) = x-1+x-2-x+3$

$= x$

$f(2) = |2-1| + |2-2| + |2-3|$

$= 2$

$LHL = \underset{x \to 2^-}{Lt} f(x)$

$= \underset{x \to 2^-}{Lt} (4-x)$

$= 2$

$RHL = \underset{x \to 2^+}{Lt} f(x)$

$= \underset{x \to 2^+}{Lt} x$

$= 2$

$\therefore LHL = RHL$

$f(x) = f(2) = 2$

$\therefore f(x)$ is continuous at $x = 2$

Continuity at $x = 3$

If $x > 1, x > 2, x < 3$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x > 2 \Rightarrow |x-2| = x-2$

If $x < 3 \Rightarrow |x-3| = -(x-3)$

$f(x) = x-1+x-2-x+3$

$= x$

If $x > 1, x > 2, x > 3$

If $x > 1 \Rightarrow |x-1| = x-1$

If $x > 2 \Rightarrow |x-2| = x-2$

If $x > 3 \Rightarrow |x-3| = x-3$

$f(x) = x-1+x-2+x-3$

$= 3x-6$

$f(3) = |3-1| + |3-2| + |3-3|$

$= 3$

$LHL = \underset{x \to 3^-}{Lt} f(x)$

$= \underset{x \to 3^-}{Lt} (x)$

$= 3$

$RHL = \underset{x \to 3^+}{Lt} f(x)$

$= \underset{x \to 3^+}{Lt} (3x-6)$

$= 9-6$

$= 3$

$\therefore LHL = RHL$

$f(x) = f(3) = 3$

$\therefore f(x)$ is continuous at $x = 3$

Theorem 1 : $\underset{x \to a}{Lt} f(x) = l$ then $\underset{x \to a}{Lt} |f(x)| = |l|$. Is converse true.

proof : suppose $\underset{x \to a}{Lt} f(x) = l$

Now we have to prove that $\underset{x \to a}{Lt} |f(x)| = |l|$

$\underset{x \to a}{Lt} f(x) = l$ $\forall \epsilon > 0$ $\exists$ $\delta > 0$ such that $|x-a| < \delta$

$\Rightarrow |f(x) - l| < \epsilon$

now $\big||f(x)| - |l|\big| < |f(x) - l|$

$\big||f(x)| - |l|\big| < \epsilon$

$\therefore$ $|f(x)|$ has limit at $x = a$

ie; $\underset{x \to a}{Lt} f(x) = |l|$

Converse part :-

By an example $f(x) = \begin{cases} 1, & x > a \\ -1, & x < a \end{cases}$

$f(x) = 1 \Rightarrow |f(x)| = 1$

$f(x) = -1 \Rightarrow |f(x)| = 1$

$\underset{x \to a}{Lt} f(x) = 1$

$\therefore$ $|f(x)|$ has limit but $\underset{x \to a}{Lt} f(x)$ does not exist.

$\therefore$ converse part is not true.

Theorem 2 : If $\underset{x \to a}{Lt} f(x) = f(a)$ then $\underset{x \to a}{Lt} |f(x)| = |f(a)|$. Is converse true.

proof : Suppose $\underset{x \to a}{Lt} = f(x) = f(a)$

Now, we have to prove that $\underset{x \to a}{Lt} |f(x)| = |f(a)|$

$\underset{x \to a}{Lt} f(x) = f(a) \Rightarrow \forall \epsilon > 0$ $\exists$ $\delta < 0$ such that

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

Now, $\left||f(x)| - |f(a)|\right| < |f(x) - f(a)|$ $\quad (||A|-|B|| < |A-B|)$

$< \epsilon$

$\therefore f(x)$ is continous at $x = a$

ie, $\underset{x \to a}{Lt} |f(x)| = |f(a)|$

<u>converse part</u> :

By an example $f(x) = \begin{cases} 1, & x \geq a \\ -1, & x < a \end{cases}$

$f(x) = 1 \Rightarrow |f(x)| = 1$

$f(x) = -1 \Rightarrow |f(x)| = 1$

Also $|f(a)| = 1$

$\underset{x \to a}{Lt} |f(x)| = |f(a)|$

$\therefore f(x)$ is continous but $\underset{x \to a}{Lt} f(x)$ is not exist.

$\therefore$ converse part is not true.

Bounded above (or) upper bounded :

An aggregate 's' is said to be bounded above if exist a real number 'k' such that $x \leq k_1, \forall x \in S$

Where k_1 = upper

If k_1 is an upper bound of 'S' then any number greater than k_1 is also upper bound of 'S'.

Least upper bound (or) suprimam :

A real number 'v' is called least upper bound of 'S' if i) 'v' is an upper bound of 'S'.

ii) A real number Less than 'v' is not upper bound of 'S'.

Bounded below (లే) lower bounded:

An aggregate 'S' is said to be bounded below if there exist a real number k_2 such that $k_2 \geq x, \forall x \in S$

Here k_2 = lower bound of 'S'.

If k_2 is lower bound of 'S' then any number less than k_2 is also lower bound of 'S'.

Greatest lower bound (లే) Infimum:

A real Number 'L' is called greatest lower bound of 'S'.

If i) 'L' is an lower bound of 'S'.

ii) Any Number greater than 'L' is not an lower bound of 'S'.

Bounded Set:

An aggregate 'S' is said to be bounded if it is both bounded above and bounded below.

(లే)

An aggregate 'S' is said to be bounded if there are two real numbers K_1 and K_2 such that $K_1 \leq x \leq K_2 \ \forall x \in S$

(లే)

A Function $f: S \to R$ is bounded on 'S' if there is a Real number 'k' such that $|f(x)| \leq k, \forall x \in S$.

Theorem 1: If 'f' and 'g' are continuous function at a then f+g is also continous at a.

proof: suppose f and g are continuous at a that is

$\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$

$\cancel{(f+g)}x \Rightarrow f(x)+g(x)$

Now, We have to prove that $\left(\lim_{x \to a} \cancel{(f+g)x = f(a)+g(a)}\right)$

$$\lim_{x \to a} (f+g)x = (f+g)a$$

$$\underset{x \to a}{Lt} (f+g)x = \underset{x \to a}{Lt} [f(x)+g(x)]$$

$$= \underset{x \to a}{Lt} f(x) + \underset{x \to a}{Lt} g(x)$$

$$= f(a) + f(g)$$

$$= (f+g)a$$

Theorem - 2 : If f and g are continuous functions at 'a' then $(f-g)$ is also continuous at 'a'.

proof: Suppose f and g are continuous at 'a'

ie, $\underset{x \to a}{Lt} f(x) = f(a)$ and $\underset{x \to a}{Lt} g(x) = g(a)$

Now, We have to prove that

$$\underset{x \to a}{Lt} (f-g)x = (f-g)a$$

$$\underset{x \to a}{Lt} (f-g)x = \underset{x \to a}{Lt} [f(x) - g(x)]$$

$$= \underset{x \to a}{Lt} f(x) - \underset{x \to a}{Lt} g(x)$$

$$= f(a) - g(a)$$

$$= (f-g)a$$

Theorem - 3 : If f and g are continuous functions at 'a' then ~~f(g)~~ (fg) is also continuous at 'a'.

proof : Suppose f and g are continuous at 'a'.

i.e. $\underset{x \to a}{Lt} f(x) = f(a)$ and $\underset{x \to a}{Lt} g(x) = g(a)$

Now, We have to prove that

$$\underset{x \to a}{Lt} (fg)x = (fg)a$$

$$\underset{x \to a}{Lt} (fg)x = \underset{x \to a}{Lt} [f(x) \cdot g(x)]$$

$$= \underset{x \to a}{Lt} f(x) \cdot \underset{x \to a}{Lt} g(x)$$

$$= f(a) \cdot g(a)$$

$$= (fg)a.$$

Theorem-4: If f and g are continuous functions at 'a' then $\left[\frac{f}{g}\right]$ is also continuous at 'a'.

proof: Suppose f and g are continuous at 'a'

ie, $\underset{x \to a}{Lt} f(x) = f(a)$ and $\underset{x \to a}{Lt} g(x) = g(a)$

Now, we have to prove that

$$\underset{x \to a}{Lt} \left[\frac{f}{g}\right]x = \left(\frac{f}{g}\right)a$$

$$\underset{x \to a}{Lt} (f/g)x = \underset{x \to a}{Lt} \left[\frac{f(x)}{g(x)}\right]$$

$$= \frac{\underset{x \to a}{Lt} f(x)}{\underset{x \to a}{Lt} g(x)}$$

$$= \frac{f(a)}{g(a)}$$

$$= \left(\frac{f}{g}\right)a$$

Imp
Theorem 5 : If 'f' is continous at 'a' & $c \in R$ is constent then of is also continous at 'a'.

proof : suppose 'f' is continous at 'a'

ie; $\underset{x \to a}{Lt} f(x) = f(a)$

Now, We prove that $\underset{x \to a}{Lt} (cf)(x) = (f(a))$

$$\underset{x \to a}{Lt} (cf)(x) = \underset{x \to a}{Lt} cf(x)$$

$$= c \underset{x \to a}{Lt} f(x)$$

$$= cf(a)$$

$\therefore$ cf is continuous at 'a'.

Theorem 6 : If 'f' is continous on $[a,b]$ then 'f' is bounded on $[a,b]$

proof : suppose 'f' is continous on $[a,b]$

Now, We prove that 'f' is bounded on $[a,b]$

for $\epsilon > 0$, We divided $[a,b]$ into a finite number of sub-intervels.

ie, $[a,b] = [a = a_0, a_1], [a_1 a_2], [a_2, a_3] \ldots\ldots [a_{n-1}. a_n = b]$

let x_1, x_2 belongs to any sub intervals such that

$$|f(x_1) - f(x_2)| < \epsilon \to ①$$

let x be any point in first sub interval $[a_0, a_1]$

By ① $|f(x) - f(a)| < \epsilon$

now, $|f(x)| = |f(x) - f(a) + f(a)|$ $[\because x \in (a,b) = a < x < b]$

$< |f(x) - f(a)| + f(a)$ $[\because x \in (a,b) = a \leq x < b]$

$\therefore$ f is particular $|f(a_1)| < \epsilon + |f(a)|$ $[\because$ By def ③$]$

In particular $|f(a_1)| < \epsilon + |f(a)|$

By ① $|f(x) - f(a_1)| < \epsilon$

now $f(x) = |f(x) - f(a_1) + f(a_1)|$

$\leq |f(x) - f(a_1)| + |f(a_1)|$

$< \epsilon + |f(a_1)|$

$< \epsilon + \epsilon + |f(a)|$

$< 2\epsilon + |f(a)|$

$\therefore$ f is bounded on $[a_1, a_2]$

In particular $|f(a_2)| < 2\epsilon + |f(a)|$

$|f(a_n)| < n\epsilon + f(a)$

$\therefore$ f is bounded on $[a_{n-1}, a_n]$

$\therefore$ f is bounded on $[a,b]$

Imp (G)

Theorem 7:

If 'f' is continuous function on $[a,b]$ then $f(x)$ attains its bounds on $[a,b]$

(or)

If 'f' is continuous Function on $[a,b]$ then $f(x)$ attains its sumprimum and Infinium on $[a,b]$

Proof: Suppose $f(x)$ is continous on $[a,b] \Rightarrow f(x)$ is bounded on $[a,b]$

$\Rightarrow f(x)$ Exists suprimum and Infinium on $[a,b]$

Let Suprimum of $f(x) = M$ & Infinium $f(x) = m \ \forall x \in [a,b]$

Now We prove that $f(x)$ attains its bounds on $[a,b]$

ie; $f(x) = M$ & $f(x) = m$, $\forall x \in [a,b]$

for this let us assume $f(x) \neq M \Rightarrow M - f(x) \neq 0$

Since $f(x)$ is continuous and constant Function (M) is also continuous

$M - f(x)$ is continuous on $[a,b]$

$\Rightarrow \dfrac{1}{M - f(x)}$ is continuous on $[a,b]$

$\Rightarrow \frac{1}{M-f(x)}$ is bounded on $[a,b]$.

$\therefore$ upper bounded and lower bounded exist.

Let 'k' be the upper bounded of $\frac{1}{M-f(x)}$

$\Rightarrow \frac{1}{M-f(x)} \leq k$

$\Rightarrow M-f(x) \geq k$

$\Rightarrow m-\frac{1}{k} \geq f(x)$ (or) $f(x) \leq m-\frac{1}{k}$ (constant)

$\therefore M-\frac{1}{k}$ is upper bounded of $f(x)$

$x \in [a,b]$
$a \leq x \leq b$

$\therefore$ M is upper bounded of $f(x)$.

$x \in (a,b)$
$a < x < b$

So our assumption $f(x) \neq M$ is wrong

$\therefore f(x) = M, \forall x \in [a,b]$

Similarly we can prove that $f(x) = m, \forall x \in [a,b]$.

$\therefore f(x)$ attains its bounded on $[a,b]$.

Theorem 8: If 'f' is continuous at $x=c$ and $f(c) \neq 0$ then show that there exists a positive number $\delta > 0$ such that $f(c)$ and $f(x)$ are with same sign. $\forall x \in (c-\delta, c+\delta)$

proof:

Suppose 'f' is continuous at $x=c$ and $f(c) \neq 0$

$\Rightarrow \forall \epsilon > 0 \; \exists \; \delta > 0$ such that $|x-c| < \delta \Rightarrow |f(x)-f(c)| < \epsilon$

$\Rightarrow c-\delta < x < c+\delta \Rightarrow f(c)-\epsilon < f(x) < f(c)+\epsilon \rightarrow (1)$

Now, we prove that $f(x)$ and $f(c)$ are same sign $\forall x \in (c-\delta, c+\delta)$

case (i): Let $f(c) > 0$ and let $\epsilon = f(c)$

then (1) $\Rightarrow f(c)-f(c) < f(x) < f(c)+f(c)$

$\Rightarrow 0 < f(x) < 2f(c)$

$\Rightarrow f(x) > 0$

$\therefore f(c)$ and $f(x)$ have same sign $\forall x \in (c-\delta, c+\delta)$

case (ii): Let $f(c) < 0$ and let $\epsilon = -f(c)$

then ① $\Rightarrow f(c) + f(c) < f(x) < f(c) - f(c)$

$\Rightarrow 2f(c) < f(x) < 0$

$\Rightarrow f(x) < 0$

$\therefore f(x)$ and $f(c)$ have same sign $\forall x \in (c-\delta, c+\delta)$

v.Imp

Q) Intermediate value theorem:

statement: If 'f' is continuous on $[a,b]$ and $f(a) \neq f(b)$ then 'f' assumes every value b/w $f(a)$ and $f(b)$ atleast once.

proof: Suppose 'f' is continuous on$[a,b]$ and $f(a) \neq f(b)$

let $f(a) < f(b)$

Let 'k' be any real number such that

$f(a) < k < f(b)$

Now, We assume that $g(x)$ on $[a,b]$ as $g(x) = f(x) - k, \forall x \in [a,b] \rightarrow$ ①

Since, $f(x)$ is continuous and constant function 'k' is also continuous.

$\therefore g(x)$ is continuous on $[a,b]$

$\therefore g(x)$ is bounded and attains its bound on $[a,b]$

From ①, $g(a) = f(a) - k$ $\quad$ $[\because f(a) < k$

$\Rightarrow g(a) < 0$ $\quad$ $f(a) - k < 0]$

From ①, $g(b) = f(b) - k$ $\quad$ $[\because k < f(b)$ (or) $f(b) > k$

$\Rightarrow g(b) > 0$ $\quad$ $\rightarrow f(b) - k > 0]$

By known theorem, $g(a) = g(b)$ have opposite sign there exists

$c \in [a,b] \Rightarrow g(c) = 0$

$\Rightarrow f(c) - k = 0$

$\Rightarrow f(c) = k$

$\therefore$ Hence the theorem proved

Definition:

uniformly continuous function:

A function $f: S \to R$ is said to be uniformly continuous on S; if $\forall\, \epsilon > 0\ \exists\ \delta > 0$ such that $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$

Theorem-10:

If $f: S \to R$ is uniformly continuous then 'f' is continuous on S.

proof: suppose 'f' is ~~con~~ uniformly continuous on S, if $\forall\, \epsilon > 0\ \exists\ \delta > 0$ such that $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon \to (1)$

Let $a \in S$

Take $x_1 = x$ and $x_2 = a$

from (1); $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

$\therefore$ 'f' is continuous on 'S'.

Theorem-11: If $f: S \to R$ is continuous on $[a,b]$ then 'f' is uniformly continuous on $[a,b]$.

proof: suppose 'f' is continuous on $[a,b]$

for $\epsilon > 0$ we divide $[a = a_0, a_1], [a_1, a_2] \ldots\ldots [a_{n-1}, a_n = b]$

$\Rightarrow$ If x_1, x_2 belongs to same sub intervals then

$|f(x_1) - f(x_2)| < \frac{\epsilon}{2} \to (1)$

let $\delta = \frac{1}{2} \min \{|a_r - a_{r-1}|$, such that $1 \le r \le n\}$

$\therefore \delta > 0$

Now, we have to P.T 'f' is uniformly continuous on $[a,b]$.

case(i): Let x_1, x_2 belongs to same interval, $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that

$|f(x_1) - f(x_2)| < \frac{\epsilon}{2} < \epsilon$

$\therefore f(x)$ is uniformly continuous on $[a,b]$.

case(ii): Let x_1, x_2 belongs to consecutive sub intervals with a_r

from (1) $|f(x_1) - f(a_r)| < \frac{\epsilon}{2}$ and $|f(a_r) - f(x_2)| < \frac{\epsilon}{2}$

$$|f(x_1) - f(x_2)| = |f(x_1) - f(a_r) + f(a_r) - f(x_2)|$$

$$< |f(x_1) - f(a_r)| + |f(a_r) - f(x_2)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon$$

$\therefore f(x)$ is uniformly continuous on $[a,b]$.